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Mixtape Chapter8 – Panel Data

(8.2.2.) Fixed effects (supplement)

$\hat{\delta}, \widehat{u_i}$ $(i = 1,2,\dots,N)$の導出

以下の回帰式を考える。

$$Y_{it} = D_{it}\delta + u_i + \varepsilon_{it} \quad \forall t \in \mathbb{Z}, \in [1,T] \quad (D_{it} = (D_{it1}, D_{it2}, \dots, D_{itk})_{1\times k}, \quad \delta = {}^t(\delta_1, \delta_2, \dots, \delta_k))$$

$$\left(\hat{\delta}, \widehat{u_1}, \dots \widehat{u_N}\right) = \arg \min_{\beta, m_1, \dots m_N} \sum_{i=1}^{N}\sum_{t=1}^{T}(Y_{it} - D_{it}\beta - m_i)^2$$

!) 解きたい最小化問題は次の通り。

$$\min \sum_{i=1}^{N}\sum_{t=1}^{T}(Y_{it} - D_{it}\beta - m_i)^2$$

FOCs :

$$\begin{cases} \frac{\partial}{\partial m_i}\Big|_{\beta=\hat{\delta},\ m_i=\widehat{u_i}} \sum_{i=1}^{N}\sum_{t=1}^{T}(Y_{it} - D_{it}\beta - m_i)^2 = 0 \quad \dots \text{(i)} \\ \frac{\partial}{\partial \beta_j}\Big|_{\beta=\hat{\delta},\ m_i=\widehat{u_i}} \sum_{i=1}^{N}\sum_{t=1}^{T}(Y_{it} - D_{it}\beta - m_i)^2 = 0 \quad (\forall j \in \mathbb{Z}, [1,k]) \quad \dots \text{(ii)} \end{cases}$$

$$\text{(i)} \dots \frac{\partial}{\partial m_i}\Big|_{\beta=\hat{\delta},\ m_i=\widehat{u_i}} \sum_{i=1}^{N}\sum_{t=1}^{T}(Y_{it} - D_{it}\beta - m_i)^2$$

$$= \sum_{t=1}^{T}\left((-2) * \left(Y_{it} - D_{it}\hat{\delta} - \widehat{u_i}\right)\right) = 0 \quad \therefore \sum_{t=1}^{T}\left(Y_{it} - D_{it}\hat{\delta} - \widehat{u_i}\right) = 0$$

$$\text{(ii)} \dots \frac{\partial}{\partial \beta_j}\Big|_{\beta=\hat{\delta},\ m_i=\widehat{u_i}} \sum_{i=1}^{N}\sum_{t=1}^{T}(Y_{it} - D_{it}\beta - m_i)^2$$

$$= \frac{\partial}{\partial \beta_j}\Big|_{\beta=\hat{\delta},\ m_i=\widehat{u_i}} \sum_{i=1}^{N}\sum_{t=1}^{T}\left(Y_{it} - \sum_{j=1}^{k} D_{itj}\beta_j - m_i\right)^2$$

$$= \sum_{i=1}^{N}\sum_{t=1}^{T}\left((2D_{itj})\left(Y_{it} - D_{it}\hat{\delta} - \widehat{u_i}\right)\right) = 0 \quad (\forall j \in \mathbb{Z}, [1,k])$$

$$\therefore \sum_{i=1}^{N}\sum_{t=1}^{T}\left((D_{itj})\left(Y_{it} - D_{it}\hat{\delta} - \widehat{u_i}\right)\right) = 0 \quad (\forall j \in \mathbb{Z}, [1,k])$$

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^t(D_{it1}, D_{it2}, \dots, D_{itk})\left(Y_{it} - D_{it}\hat{\delta} - \widehat{u_i}\right)\right) = 0$$

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}D_{it}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_\iota}\right)\right)=0$$

これゆえ、一階条件は以下の様に書き表される:

$$\begin{cases} \sum_{t=1}^{T}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_\iota}\right)=0 \\ \sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}D_{it}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_\iota}\right)\right)=0 \end{cases}$$

$\therefore\ for\ i=1,2,\ldots,N\ ;\ldots$

$$\begin{cases} \widehat{u_\iota}=\frac{1}{T}*\sum_{t=1}^{T}\left(Y_{it}-D_{it}\hat{\delta}\right)=\overline{Y_\iota}-\overline{D_\iota}\hat{\delta} \quad \left(\overline{Y_\iota}\equiv\frac{1}{T}*\sum_{t=1}^{T}Y_{it}\ ;\ \overline{D_\iota}\equiv\frac{1}{T}*\sum_{t=1}^{T}D_{it}\right) \\ \sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}D_{it}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_\iota}\right)\right)=0 \end{cases} \ldots(*)$$

ここで、

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}\overline{D_\iota}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_\iota}\right)\right)=0$$

であることを示す。(☆あとでコレ↑を使います。)

$$\overline{D_\iota}\equiv\frac{1}{T}*\sum_{t=1}^{T}D_{it}$$

であり、これより

$${}^{t}\overline{D_\iota}\equiv\frac{1}{T}*\sum_{t=1}^{T}{}^{t}D_{it}$$

が成立(※脚注参照。)[1]　これを用いて、

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$${}^{t}\overline{D_\iota}\equiv{}^{t}\left(\frac{1}{T}*\sum_{t=1}^{T}D_{it}\right)={}^{t}\left(\frac{1}{T}*\sum_{t=1}^{T}\left(D_{it1}\ \ D_{it2}\ \ \ldots\ \ D_{itk}\right)_{1\times k}\right)$$

$$={}^{t}\left(\frac{1}{T}\sum_{t=1}^{T}D_{it1}\ \ \frac{1}{T}\sum_{t=1}^{T}D_{it2}\ \ \ldots\ \ \frac{1}{T}\sum_{t=1}^{T}D_{itk}\right)$$

他方、

$$\frac{1}{T}*\sum_{t=1}^{T}{}^{t}D_{it}=\frac{1}{T}*\sum_{t=1}^{T}{}^{t}\left(D_{it1}\ \ D_{it2}\ \ \ldots\ \ D_{itk}\right)_{1\times k}$$

$$={}^{t}\left(\frac{1}{T}\sum_{t=1}^{T}D_{it1}\ \ \frac{1}{T}\sum_{t=1}^{T}D_{it2}\ \ \ldots\ \ \frac{1}{T}\sum_{t=1}^{T}D_{itk}\right)$$

依って、

$${}^{t}\overline{D_\iota}\equiv\frac{1}{T}*\sum_{t=1}^{T}{}^{t}D_{it}$$

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}\overline{D}_{\iota}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_{\iota}}\right)\right)=\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\frac{1}{T}*\sum_{t=1}^{T}{}^{t}D_{it}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_{\iota}}\right)\right)$$

$$=\sum_{i=1}^{N}\sum_{t=1}^{T}\frac{1}{T}*\left(\sum_{t=1}^{T}{}^{t}D_{it}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_{\iota}}\right)\right)\ldots(+)$$

ここで、$\sum_{t=1}^{T}{}^{t}D_{it}*\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_{\iota}}\right)$は$t$に依らないことから、

$$(+)=\sum_{i=1}^{N}\frac{1}{T}*\sum_{t=1}^{T}\left(\sum_{t=1}^{T}{}^{t}D_{it}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_{\iota}}\right)\right)=\sum_{i=1}^{N}\frac{1}{T}*T*\sum_{t=1}^{T}{}^{t}D_{it}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_{\iota}}\right)$$

$$=\sum_{i=1}^{N}\sum_{t=1}^{T}{}^{t}D_{it}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_{\iota}}\right)=0\quad(\because (*)\text{ の第二式に依る})$$

よって、

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}\overline{D}_{\iota}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_{\iota}}\right)\right)=0\quad\ldots(**)$$

が示された。

今、(*)を書き直すと、

$$for\ i=1,2,\ldots,N\begin{cases}\widehat{u_{\iota}}=\dfrac{1}{T}*\displaystyle\sum_{t=1}^{T}\left(Y_{it}-D_{it}\hat{\delta}\right)=\overline{Y}_{\iota}-\overline{D}_{\iota}\hat{\delta}\quad\left(\overline{Y}_{\iota}=\dfrac{1}{T}*\displaystyle\sum_{t=1}^{T}Y_{it};\ \overline{D}_{\iota}=\dfrac{1}{T}*\displaystyle\sum_{t=1}^{T}D_{it}\right)\\ \displaystyle\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}D_{it}\left(Y_{it}-D_{it}\hat{\delta}-\left(\overline{Y}_{\iota}-\overline{D}_{\iota}\hat{\delta}\right)\right)\right)=0\ldots(\mathrm{iii})\end{cases}$$

$$(\mathrm{iii})\Leftrightarrow\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}D_{it}\left(Y_{it}-\overline{Y}_{\iota}-D_{it}\hat{\delta}+\overline{D}_{\iota}\hat{\delta}\right)\right)=0$$

$$\Leftrightarrow\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}\overline{D}_{\iota}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_{\iota}}\right)\right)-0=0-0$$

$$\Leftrightarrow\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}D_{it}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_{\iota}}\right)\right)-\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}\overline{D}_{\iota}\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_{\iota}}\right)\right)=0\quad(\because(**))$$

$$\Leftrightarrow\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\left({}^{t}D_{it}-{}^{t}\overline{D}_{\iota}\right)\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_{\iota}}\right)\right)=\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}(D_{it}-\overline{D}_{\iota})\left(Y_{it}-D_{it}\hat{\delta}-\widehat{u_{\iota}}\right)\right)=0$$

$$\Leftrightarrow\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}(D_{it}-\overline{D}_{\iota})\left(Y_{it}-D_{it}\hat{\delta}-\left(\overline{Y}_{\iota}-\overline{D}_{\iota}\hat{\delta}\right)\right)\right)=0$$

$$\Leftrightarrow\sum_{i=1}^{N}\left(\sum_{t=1}^{T}\left({}^{t}(D_{it}-\overline{D}_{\iota})(Y_{it}-\overline{Y}_{\iota})\right)-\sum_{t=1}^{T}\left({}^{t}(D_{it}-\overline{D}_{\iota})\left(D_{it}\hat{\delta}-\overline{D}_{\iota}\hat{\delta}\right)\right)\right)$$

$$=\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}(D_{it}-\overline{D}_{\iota})(Y_{it}-\overline{Y}_{\iota})\right)-\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}(D_{it}-\overline{D}_{\iota})\left(D_{it}\hat{\delta}-\overline{D}_{\iota}\hat{\delta}\right)\right)=0$$

$$\therefore \sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}(D_{it}-\overline{D_{\iota}})(Y_{it}-\overline{Y_{\iota}})\right) = \sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}(D_{it}-\overline{D_{\iota}})(D_{it}-\overline{D_{\iota}})\right) * \hat{\delta}$$

ここで、$\ddot{D_{\iota t}} \equiv D_{it}-\overline{D_{\iota}}$, $\ddot{Y_{\iota t}} \equiv Y_{it}-\overline{Y_{\iota}}$ と置くと、（※Mixed Tape の表記は恐らく誤植です。）

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}\ddot{D_{\iota t}}\ddot{Y_{\iota t}}\right) = \sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}\ddot{D_{\iota t}}\ddot{D_{\iota t}}\right) * \hat{\delta}$$

$\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}\ddot{D_{\iota t}}\ddot{D_{\iota t}}\right)$は行列であることに注意して、

$$\hat{\delta} = \left(\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}\ddot{D_{\iota t}}\ddot{D_{\iota t}}\right)\right)^{-1}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}\ddot{D_{\iota t}}\ddot{Y_{\iota t}}\right)\right)$$

以上をまとめると、

$$\begin{cases} \widehat{u_{\iota}} = \overline{Y_{\iota}} - \overline{D_{\iota}}\hat{\delta} \\ \hat{\delta} = \left(\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}\ddot{D_{\iota t}}\ddot{D_{\iota t}}\right)\right)^{-1}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\left({}^{t}\ddot{D_{\iota t}}\ddot{Y_{\iota t}}\right)\right) \end{cases}$$

が本回帰式において求める推定値となる。■